The compact action realization problem

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Introduction

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Realizations of countable Borel equivalence relations, 5/11/2023

Introduction

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The theory of **countable Borel equivalence relations (CBER)** has been a very active area of research in descriptive set theory during the last three decades and has many interesting connections with other areas of mathematics, such as dynamical systems, operator algebras, geometric group theory, combinatorics and probability.

A CBER is a Borel equivalence relation on a standard Borel space (i.e., a Polish space with its Borel structure) all of whose equivalence classes are countable, e.g., the equivalence relation induced by a Borel action of a countable group. In fact by a theorem of Feldman and Moore all CBER are indeed induced by Borel actions of countable groups.

Theorem (Feldman-Moore, 1977)

If E is a CBER on a standard Borel space X, then there is a countable group G and a Borel action of G on X which induces E, i.e., the E-classes are the orbits of the action.

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So in principle all CBER come from such actions. However there are many important examples of CBER that are not naturally defined via actions.

To avoid uninteresting from our point of view situations, unless it is otherwise explicitly stated or clear from the context, all the standard Borel or Polish spaces below will be uncountable and all CBER *will be* **aperiodic**, *i.e., have infinite classes.*

Introduction

Given CBER E, F on standard Borel spaces X, Y, resp., a Borel isomorphism of E with F is a Borel bijection $f\colon X \to Y$ which takes E to F. If such f exists, we say that E, F are **Borel isomorphic**, in symbols $E \cong_B F$. Generally speaking a **realization** of a CBER E is a CBER $F \cong_B E$ with desirable properties.

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To start with, the Feldman-Moore Theorem implies (via some basic descriptive set theory) that any CBER has a realization which is induced by a continuous action of a countable (discrete) group on a Polish space. We call these **continuous action realizations** and we would like to understand such realizations for which the space and the action have additional properties.

This study has many different aspects but in this talk I will focus on one important problem, the problem of realization by a continuous action on a compact Polish space. A **compact action realization** of a CBER E is a realization F of E that is induced by a continuous action of countable group on a compact Polish space. Such an F is always non-smooth, where a **smooth** CBER is one which admits a Borel transversal, so the basic problem is the following:

This study has many different aspects but in this talk I will focus on one important problem, the problem of realization by a continuous action on a compact Polish space. A **compact action realization** of a CBER E is a realization F of E that is induced by a continuous action of countable group on a compact Polish space. Such an F is always non-smooth, where a **smooth** CBER is one which admits a Borel transversal, so the basic problem is the following:

Problem (The compact action realization problem)

Does every non-smooth CBER *admit a compact action realization?*

Introduction

I will devote the first part of my talk to recent results concerning this problem. The second part of the talk will concentrate on questions concerning spaces of subshifts, since subshifts play an important role in this study. For each countable group Γ and topological space X, consider the shift action of Γ on X^Γ defined by

$$\gamma \cdot p(\delta) = p(\gamma^{-1}\delta), \gamma, \delta \in \Gamma, p \in X^\Gamma.$$

A **subshift** of X^Γ is the restriction of this action to a nonempty invariant closed subset of X^Γ.

CBER

Before I start discussing results about the compact action realization problem, let's look at the global picture of CBER. One usually organizes CBER under the hierarchical order of Borel reducibility. A CBER E on X is **Borel reducible** to a CBER F on Y, in symbols

$$E \leq_B F,$$

if there is a Borel function $f\colon X \to Y$ such that

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$

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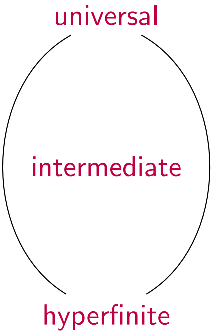
Then $\leq_B$ is pre-order on CBER and among non-smooth CBER it looks as follows:

CBER



There is a **minimum** element consisting of the **hyperfinite** CBER (i.e., those that are increasing unions of a sequence of CBER with finite classes or equivalently induced by Borel actions of the group $\mathbb{Z}$) a typical example of which is the CBER induced by the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$. There is also a **maximum** element consisting of the **universal** ones, a typical example of which is the CBER induced by the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$.

There is also a vast number of **intermediate** ones. These include for example the CBER induced by the free part of the shift action of Γ on 2^{Γ}, for any non-amenable group Γ. Freeness is crucial here since no free action can generate a universal CBER by a result of Simon Thomas (that also uses Popa's cocycle superrigidity theory).

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Theorem

Every (non-smooth aperiodic) hyperfinite CBER E *admits a compact action realization. In fact this can be taken to be a subshift of* $2^{\mathbb{F}_2}$ *if* E *is compressible and a subshift of* $2^{\mathbb{Z}}$ *otherwise.*

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A CBER E on X is **compressible** if there is a Borel injection $f\colon X \to X$ with $f(C) \subsetneqq C$, for every E-class C. By a theorem of Nadkarni this is equivalent to the non-existence of an invariant probability Borel measure for E.

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For each infinite countable group Γ, let $E(\Gamma, 2^{\mathbb{N}})$ be the equivalence relation induced by the shift action of Γ on $(2^{\mathbb{N}})^{\Gamma}$. Let $Ap((2^{\mathbb{N}})^{\Gamma})$ be the **aperiodic part** of $(2^{\mathbb{N}})^{\Gamma}$, i.e., the set of points x with infinite orbit, and let $E^{ap}(\Gamma, 2^{\mathbb{N}})$ be the restriction of $E(\Gamma, 2^{\mathbb{N}})$ to $Ap((2^{\mathbb{N}})^{\Gamma})$. Let also $Fr((2^{\mathbb{N}})^{\Gamma})$ be its **free part**, i.e., the set of points x such that $\gamma \cdot x \neq x, \ \forall \gamma \in \Gamma, \gamma \neq 1$. Denote by $F(\Gamma, 2^{\mathbb{N}})$ the restriction of $E(\Gamma, 2^{\mathbb{N}})$ to $Fr((2^{\mathbb{N}})^{\Gamma})$.

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Recall that if Γ is not amenable these CBER are intermediate.

Universal

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Theorem

(i) Every compressible, universal CBER *admits a compact action realization. In fact such a realization can be taken to be a subshift of* $2^{\mathbb{F}_4}$*.*
(ii) If Γ *is infinite and also finitely generated, then* $E^{ap}(\Gamma, 2^{\mathbb{N}})$ *admits a compact action realization, which can be taken to be a subshift of* $(2^{\mathbb{N}})^{\Gamma}$*.*

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It is known here that if Γ contains a non-abelian free group then $E^{ap}(\Gamma, 2^{\mathbb{N}})$ is universal (and incompressible).

Arithmetical vs Turing equivalence

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Since arithmetical equivalence is compressible and universal (by a result of Slaman and Steel), it follows that it admits a compact action realization. However it is a long-standing open problem whether Turing equivalence (which is also compressible) is universal and the following is open:

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Does Turing equivalence admit a compact action realization?

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Fix a countable group Γ. For any Polish space X, define the standard Borel space of subshifts of X^Γ, with the Effros Borel structure, as follows:

$$\mathrm{Sh}(\Gamma, X) = \{F \colon F \subseteq X^\Gamma \text{is closed and } \Gamma\text{−invariant}\}$$

If X is compact, we view this as a compact Polish space with the Hausdorff metric topology.

Consider the Hilbert cube $\mathbb{I}^{\mathbb{N}}$, where $\mathbb{I} = [0,1]$. Every compact Polish space is (up to homeomorphism) a closed subspace of $\mathbb{I}^{\mathbb{N}}$, and thus every **$\boldsymbol{\Gamma}$-flow** (i.e., a continuous action of Γ on a compact Polish space) is (topologically) isomorphic to a subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$. We can thus consider the compact Polish space $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ as the universal space of Γ-flows.

Subshifts as a universal space for equivalence relations

Similarly consider the product space $\mathbb{R}^{\mathbb{N}}$. Every Polish space is (up to homeomorphism) a closed subspace of $\mathbb{R}^{\mathbb{N}}$, and thus every continuous Γ-action on a Polish space is (topologically) isomorphic to a subshift of $(\mathbb{R}^{\mathbb{N}})^{\Gamma}$. We can thus consider the standard Borel space $\mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ as the universal space of continuous Γ-actions on Polish spaces.

Subshifts as a universal space for equivalence relations

In particular taking $\Gamma = \mathbb{F}_\infty$, the free group with a countably infinite set of generators, we see that every CBER is Borel isomorphic to the equivalence relation E_F induced on some subshift F of $(\mathbb{R}^\mathbb{N})^{\mathbb{F}_\infty}$ and so we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^\mathbb{N})$ also as the universal space of CBER and study the descriptive complexity of various classes of CBER (like, e.g., smooth, aperiodic, compressible, hyperfinite, etc.) as subsets of this universal space. Similarly we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^\mathbb{N})$ as the universal space of CBER that admit a compact action realization. In this case we can also consider complexity questions as well as genericity questions of various classes.

Subshifts as a universal space for equivalence relations

Problems of the descriptive complexity of various classes of CBER have been around for many years but they were formulated in terms of "codes" for Borel equivalence relations. The use of these universal spaces provides a conceptual advantage, since these can now be formulated as problems of descriptive complexity of sets in standard Borel or compact Polish spaces and in the second case also genericity problems can be considered that make no sense in the previous framework.

Let Φ be a property of CBER which is invariant under Borel isomorphism. Let

$$\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^{\mathbb{N}}) = \{F \in \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}}) : E_F \models \Phi\},$$

where we write $E_F \models \Phi$ to mean that E_F has the property Φ. Similarly we define

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In our paper we have studied the descriptive complexity and genericity properties of the sets $\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N}), \mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$, for various properties Φ and groups Γ but I will concentrate, for reasons to be explained later, in the property of hyperfiniteness and also in the case of the free groups $\Gamma = \mathbb{F}_n, 2 \leq n \leq \infty$ (although these results hold for much more general classes of countable groups).

Below a set in a Polish space is $\mathbf{\Pi}^1_1$ (or **coanalytic**), if it is the complement of an analytic set (which is the continuous image of a Borel set). A set is $\mathbf{\Sigma}^1_2$ if it is a continuous image of a $\mathbf{\Pi}^1_1$ set. $\mathbf{\Sigma}^1_2$-**complete** sets are the most complicated sets in this class and they are therefore not $\mathbf{\Pi}^1_1$ (and so not Borel).

Subshifts as a universal space for equivalence relations

We now have the following results, where a CBER is **measure hyperfinite** if it is hyperfinite μ-a.e., for every probability Borel measure μ. This is also equivalent to being **measure amenable** by the theorem of Connes-Feldman-Weiss.

Theorem

(i) The set of **free hyperfinite** *subshifts in* $\mathrm{Sh}(\mathbb{F}_n, \mathbb{I}^{\mathbb{N}})$ *is* $\boldsymbol{\Sigma}^1_2$ *(but we do not know if it is anything simpler). However the class of* **free measure hyperfinite** *subshifts in* $\mathrm{Sh}(\mathbb{F}_n, \mathbb{I}^{\mathbb{N}})$ *is* G_δ.
(ii) The class of **free measure hyperfinite** *subshifts in* $\mathrm{Sh}(\mathbb{F}_n, \mathbb{I}^{\mathbb{N}})$ *is comeager.*
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Is every CBER induced by a Borel action of a countable amenable group hyperfinite?

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A lot of work over the last few decades has provided a positive answer for many classes of amenable groups but it is still open for arbitrary amenable groups.

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One interesting possible approach to providing a negative answer is based on definability considerations. We have seen that the set of free measure hyperfinite subshifts in $\mathrm{Sh}(\mathbb{F}_n, \mathbb{I}^{\mathbb{N}})$ form a $\boldsymbol{G_\delta}$ set, while those that are free hyperfinite form a $\boldsymbol{\Sigma^1_2}$ set. If one could show that this is actually $\boldsymbol{\Sigma^1_2}$-complete or in fact much less that that, namely that it is not $\boldsymbol{G_\delta}$, this would show that the two notions are distinct, so there are measure hyperfinite CBER which are not hyperfinite.

Genericity of hyperfiniteness

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I will outline below the main idea of the proof (which is quite technical).
It is not easy to work directly with subshifts, so first one transfers the problem in a different context using the so-called Correspondence Theorem of Hochman.

Genericity of hyperfiniteness

Denote by $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ the Polish space of continuous actions of a countable group Γ on the Cantor space $2^{\mathbb{N}}$.

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Theorem (Hochman, 2019)

Let Φ be a property of continuous actions of a countable group Γ on compact Polish spaces, which is invariant under topological isomorphism. Then the following are equivalent:
(i) $\mathrm{Sh}_{\Phi}(\Gamma, \mathbb{I}^{\mathbb{N}})$ *is comeager;*
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(i) $\mathrm{Sh}_{\Phi}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is comeager;
(ii) $\mathrm{Act}_{\Phi}(\Gamma, 2^{\mathbb{N}})$ is comeager.

Thus it is enough to show that generically an action of $\mathbb{F}_n$ on the Cantor space gives a hyperfinite CBER (genericity of freeness has been known earlier). This is based on the important fact that, for finite n, generically there is, in the sense to be explained below, a unique such action!

Akin, Hurley and Kennedy asked in 2003 whether there is a generic homeomorphism of $2^{\mathbb{N}}$, i.e., a homeomorphism whose conjugacy class (under the homeomorphism group of $2^{\mathbb{N}}$) is comeager. A positive answer was proved in 2007 by K-Rosendal. Thus generically there is a unique (up to conjugacy) homeomorphism. This was extended later to actions of any finitely generated free group by Ola Kwiatkowska.

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Theorem (Kwiatkowska, 2012)

For each finite n, generically there is a unique (up to conjugacy) continuous action of $\mathbb{F}_n$ on $2^{\mathbb{N}}$.

Thus it is enough to show that this generic action is hyperfinite. Kwiatkowska's proof used the so-called **projective Fraïssé theory** of Irwin and Solecki, which also gives a method for showing that continuous actions of $\mathbb{F}_n$ that are built as certain projective limits of finite combinatorial structures are factors of the generic action. One finally shows that the boundary action of $\mathbb{F}_n$ satisfies these conditions and using the fact that this boundary action is hyperfinite, one can lift this up to show that this is also the case for the generic action.

Genericity of hyperfiniteness

This proof does not work for the infinitely generated free group $\mathbb{F}_\infty$, since this group does not have a generic action (K-Rosendal). However with additional work one can use the finitely generated case to also show genericity even for $\mathbb{F}_\infty$.

But this is not the end of the story. We know that genericity of measure hyperfiniteness is also true not only for the subshifts of free groups but much more generally for the subshifts of all **exact groups**, i.e., those that admit an amenable flow, which contain all amenable and free groups among many others. But the following is open:

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Problem

Is generically a subshift in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ hyperfinite, for all exact groups or even all amenable groups Γ?

Thank you!